Irrelevance of spatial correlations in models with extremal dynamics

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(Received 11 November 1996)

The relevance of spatial correlations set up in the quenched disorder by extremal dynamics is studied both analytically and by numerical simulations. We find that these correlations, although present in systems of small size L, vanish in the thermodynamic limit. [S1063-651X(97)01105-7]

PACS number(s): 02.50.−r, 05.40.+j, 05.90.+m

Self-organized critical models [1] with extremal dynamics have attracted significant attention recently. Many physical phenomena belong to this class of models: fluid displacement in porous disordered media [2], interface depinning [3], punctuated biological evolution [1]. In these models, at each time step the dynamical activity is concentrated on the site with the extremal value of the quenched disorder. This rule leads to a rich and complex behavior, which has been widely studied [4].

Recently, we introduced a theoretical approach for the study of these models [5–7]. This method is based on a quenched-stochastic transformation [5], also called run time statistics (RTS), which maps the extremal dynamics onto a stochastic process characterized by a probability distribution for the elementary dynamical events. This mapping makes possible the application of a real space technique, such as the fixed scale transformation [8] or the real space renormalization group [7], for the analysis of the self-organized critical properties of the models and the computation of their critical exponents. The RTS, as well as other theoretical approaches [4], assumes that the quenched variables representing the disorder of the system are independent.

In fact, this assumption seems to be only an approximation [6,9], at least for finite system size L. In this paper we perform a theoretical and numerical analysis of the relevance of correlations in problems with extremal dynamics, proving that correlations among disorder variables vanish in the thermodynamic (infinite size) limit.

The basic idea of the RTS is to encode the effects of disorder into an *effective*, time dependent, probability density of the quenched variables (for details see [5,6]). Extremal dynamics, as in invasion percolation, is based on the choice of the smallest disorder variable among those which are eligible for growth (which we shall call *active* variables hereafter). In invasion percolation, a variable becomes active only when it is ''touched'' by the growing cluster. The past growth history of extremal dynamics builds up memory effects in the system. These can be encoded into the distribution $p_t(x_1,x_2,\ldots,x_n)$ of the active variables, which acquires an explicit time dependence. The crucial point, which we are going to address here, is that it is assumed that the collective probability density of the disorder at any time t factorizes:

$$p_t(x_1,x_2,\ldots,x_n)=\prod_{i=1}^{n}\rho_{i,t}(x_i). \tag{1}$$

Before entering the detailed discussion of this approximation, it is worthwhile spending some more words on the nature of the time dependence induced by extremal dynamics. Let us assign an age $\tau_i=t-t_0$ to the variable x_i. Here t is the actual time and t_0 is the time at which the variable x_i became active. Clearly, variables that became active at the same time t_0 and which have the same age τ experienced the same history. Variables which experienced the same history have the same effective probability densities. For this reason, we can express the single variable probability density $\rho_{i,t}(x)$ in terms of the age τ_i of the active variable x_i: $\rho_{i,t}(x)=p_{\tau_i,t}(x)$.

In general Eq. (1) is not exact. Indeed, extremal dynamics sets up long range spatial correlations between quenched variables and the distribution of disorder is no longer factorizable. We shall see, however, that these correlations, although relevant for small system sizes, *vanish in the thermodynamic limit*. Usually, one is interested in the critical properties of extremal models in the thermodynamic limit. Therefore our results support the validity of the assumption of independence made in the RTS, when one is dealing with large scale systems. Our analysis does not exclude the presence of short range correlations, but they are irrelevant in the scale invariant regime.

An elementary way to compute the spatial correlations between variables set up by extremal dynamics is the following. Let us consider L independent uniform variable ϵ_i. Then, let us impose on them the condition $\epsilon_i\geqslant\epsilon_1$ for all $i=2,\ldots,L$. This condition is, e.g., imposed in the first time step of the extremal dynamics for the Bak and Sneppen (BS) model. With the condition $\epsilon_i\geqslant\epsilon_1$, the distribution of the ϵ_i is no longer that of independent variables. A straightforward calculation shows that the (marginal) distribution of one of the variables is

$$p_1(x)=\frac{L}{L-1}[1-(1-x)^{L-1}]$$

while that of a pair of variables is

$$p_2(x,y)=\frac{L+1}{L-1}\{\theta(y-x)[1-(1-x)^L]+\theta(x-y)\times[1-(1-y)^L]\}.$$

This is clearly not factorizable. However, it is easy to check that the correlation

$$\langle \epsilon_i\epsilon_j\rangle-\langle\epsilon_i\rangle\langle\epsilon_j\rangle=\frac{L}{4(L+1)^2(L+2)}\sim\frac{1}{L^2} \tag{2}$$

vanishes as $L\to\infty$. This suggests that the corrections to Eq. (1) vanish in the limit $L\to\infty$.

If, already at the first time step, the dynamics sets up a correlation between variables, it is compelling to show that the correlation remains small as $L\to\infty$. Note that L in models like invasion percolation in radial geometry is the size of the cluster which equals the time, $t=L$, so that the limit $L\to\infty$ is the asymptotic time limit in this case. Let us now try to extend our analysis to the correlation between two variables ϵ_1 and ϵ_2 with age τ, for a system of fixed size L. Consider two variables ϵ_1 and ϵ_2 with age τ for an invasion percolation cluster of size L. Imagine to rank order the L variables from the smallest $\min\epsilon_i$ to the largest $\max\epsilon_i$. Let k_1 and k_2 be the ranking of ϵ_1 and ϵ_2, respectively. One has that $k_i>\tau$ for $i=1,2$ because at least τ uniform variables, among the L, were found to be smaller than ϵ_1 and ϵ_2. We want to compute the distribution of ϵ_i.

The distribution of the kst among L uniform variables is

$$\Theta_k(x)=L\binom{L-1}{k-1}x^{k-1}(1-x)^{L-k}.$$

Assuming that the order of the variables ϵ_i can be any between $\tau+1$ and L (with equal probability), one might say that

$$p_\tau(x)=\frac{1}{L-\tau}\sum_{k=\tau+1}^{L}\Theta_k(x)$$

is the distribution of the ϵ_i. In this approximation, one finds

$$\langle\epsilon_i\rangle_\tau=\int_0^1 p_\tau(x)x\,dx=\frac{L+\tau+1}{2(L+1)}.$$

Then we can address the question of what is the joint distribution of two variables with age τ. First we recall that the joint distribution of the kth and the jth variable ($j>k$) is [10]

$$\Theta_{k,j}(x,y)=\frac{\theta(y-x)\Gamma(L+1)}{\Gamma(k)\Gamma(j-k)\Gamma(L-j+1)}\times x^{k-1}(y-x)^{j-k-1}(1-y)^{L-j},$$

where x is the value assumed by the kth variable and y that of the jth.

The joint distribution of ϵ_1 and ϵ_2 can again be computed by assuming that, apart from being bigger than τ, k and j can be any index, and therefore

$$\tilde{p}_\tau(x,y)=\frac{1}{L-\tau-1}\sum_{k=\tau+1}^{L-1}\frac{1}{L-k}\sum_{j=k+1}^{L}\Theta_{k,j}(x,y)\quad\text{for } x<y,$$

whereas for $x>y$, $\tilde{p}_\tau(y,x)$ is given by the same formula with $\Theta_{k,j}(y,x)$.

Now one can calculate the value of $\langle\langle(\epsilon_1-\langle\epsilon_1\rangle)(\epsilon_2-\langle\epsilon_2\rangle)\rangle\rangle$ and therefore the correlation buildup between two variables in τ steps of the extremal dynamics. One has

$$\langle x\rangle_k=\int_0^1\theta_k(x)x\,dx=\frac{k}{L+1},$$

$$\langle xy\rangle_{k,j}=\int_0^1\int_0^y\Theta_{k,j}(x,y)xy\,dx\,dy=\frac{k(j+1)}{(L+1)(L+2)},$$

then

$$\sigma_{k,j}=\langle xy\rangle_{k,j}-\langle x\rangle_k\langle y\rangle_j=\frac{k(L-j+1)}{(L+2)(L+1)^2}.$$

Then the correlation is the average of this for $k>\tau$ and $j>k$:

$$\sigma_\tau=\frac{1}{L-\tau-1}\sum_{k=\tau+1}^{L-1}\frac{1}{L-k}\sum_{j=k+1}^{L}\sigma_{k,j}=\frac{L^2+L\tau+2\tau-2\tau^2+4L}{12(L+2)(L+1)^2}\sim\frac{1}{L}. \tag{3}$$

Therefore one expects a correlation of order $1/L$ among two variables of the system.

In order to support this result, we also performed numerical simulations. It is worth pointing out that correlations built up by extremal dynamics are global in the sense that they depend only on the ages τ_i of the variables and not on their spatial position. The persistence of activity in the dynamics of a particular model, however, builds a particular spatial distribution of age variables and this yields a spatial correlation. Therefore correlations can be investigated directly by studying the spatial correlation among variables.

We performed numerical simulations of the simplest model with extremal dynamics, the Bak and Sneppen model in one dimension, and studied the behavior of spatial correlations for different sizes L of the system. We follow the method given in Ref. [11], which is able to detect the presence of very weak long range (power law) correlations between the variables x_i. We study the function

$$\sigma(r)=\sum_{i=1}^{r}(x_i-\overline{x}), \tag{4}$$

where r is the distance (in lattice units) between the first and the last variable, and $\overline{x}=1/L\Sigma_{i=1}^{L}x_i$ is the mean value of the quenched disordered variables. For variables with no long range correlations (this is the case of quenched variables at time $t=0$), one has

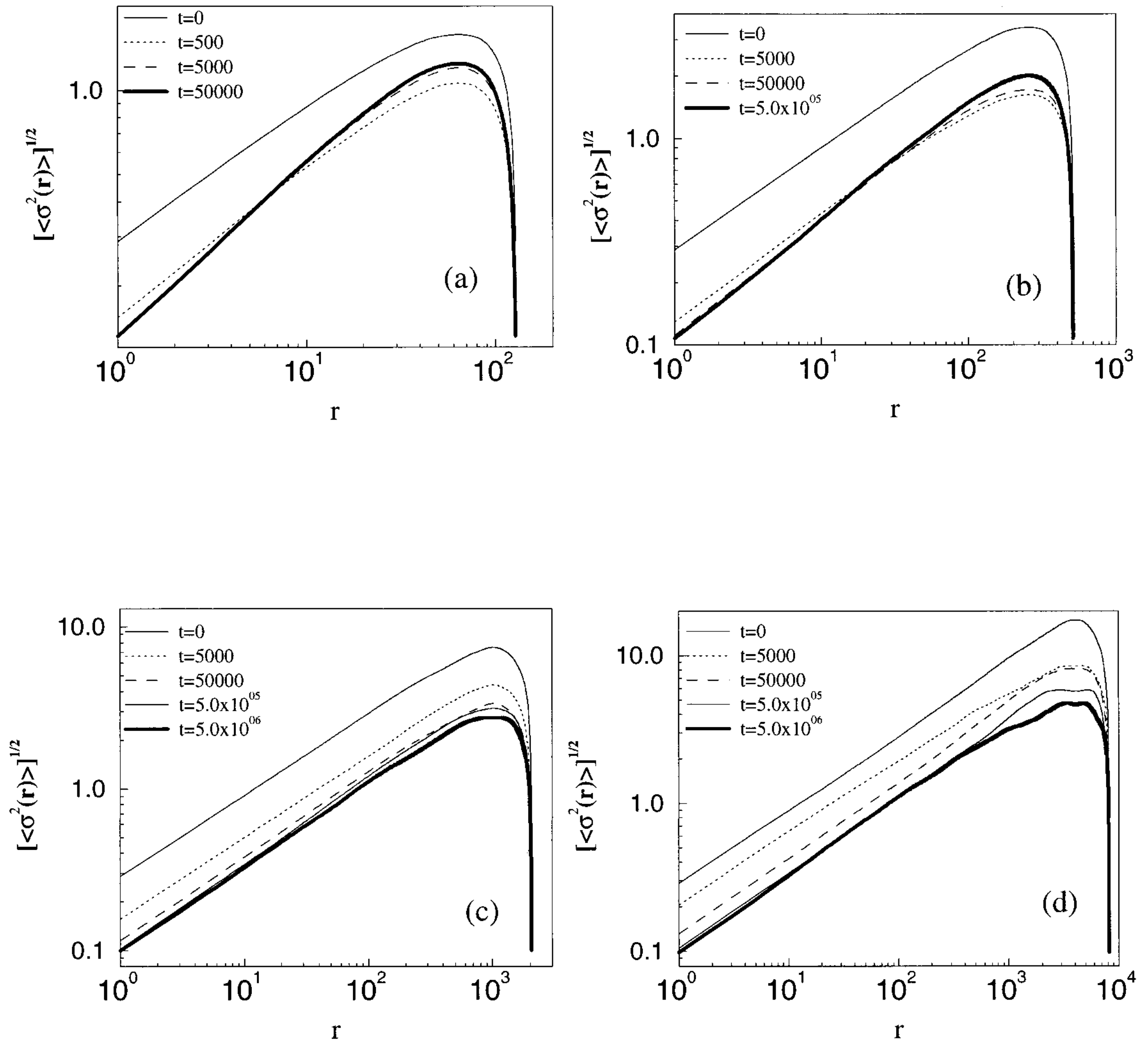

FIG. 1. The temporal evolution of the function $(\langle\sigma(r)^2\rangle)^{1/2}$ in the BS model, for some values of the system size: (a) $L=128$; (b) $L=512$; (c) $L=2048$; (d) $L=8192$. The time ranges from $t=0$ (no long range correlations) to a time step corresponding to the asymptotic critical regime.

$$[\langle\sigma(r)^2\rangle]^{1/2}\sim r^{\beta} \quad \text{with} \quad \beta=0.5, \tag{5}$$

where the mean is over the realizations of the disorder. Any deviation from 0.5 in the scaling exponent β shows the presence of long range spatial correlations between quenched variables. We performed numerical simulations of the Bak and Sneppen model for $L=$ 128,512,2048,8192. For each value of L we computed the function $(\langle\sigma_t(r)^2\rangle)^{1/2}$ for different times, from $t=0$ (no long range correlations) to a time step corresponding to the stationary state of the system (it varies depending on the size of the system; typically it goes up to $t=5.0\times10^6$ for $L=2048{,}8192$). To get a better statistics we mediate $\sigma_t(r)^2$ over all the possible starting points r_0. In Figs. 1(a)–1(d) we show the behavior of $(\langle\sigma_t(r)^2\rangle)^{1/2}$ at different times for the different sizes of the system. The falling down of $(\langle\sigma(r)^2\rangle)^{1/2}$ for large values of r is not surprising, because one sees from Eq. (4) that $\sigma_t(r=L)=0$. It is a finite size effect. We note that $(\langle\sigma(r)^2\rangle)^{1/2}$ changes in time until it achieves a steady state. We computed by a least squares fit the scaling exponent of the small r part of $(\langle\sigma(r)^2\rangle)^{1/2}$ for $t=0$ (as a test of the validity of the method) and in the steady state, for the different values of L. In Table I we show the value of the exponent β at $t=0$ and in the critical steady state for different sizes of the system. As one can see, while the value of β for the $t=0$ (no long range correlations) case is stable as the size L of the system grows, the values of the scaling exponent in the steady state, where extremal dynamics built up correlations between the variables, seem to converge in the limit $L\rightarrow\infty$ toward the value $\beta=0.5$. So, from numerical simulations we get

TABLE I. Values of the scaling exponent β, for $L=128{,}512{,}2048{,}8192$ at time $t=0$ (no long range correlations) and in the steady state.

L	$\beta_{t=0}$	β_{steady}
128	0.48 ± 0.02	0.62 ± 0.02
512	0.50 ± 0.02	0.58 ± 0.02
2048	0.50 ± 0.01	0.54 ± 0.01
8192	0.50 ± 0.01	0.50 ± 0.01

$$\lim_{L\to\infty} \beta_{\text{steady}}(L) = 0.5, \tag{6}$$

which supports qualitatively our analytical results. Short range spatial correlations could eventually be present in our extremal model, but they disappear after a rescaling of the system. Therefore they are irrelevant for the critical properties.

Our results fit those of Ref. [4]. There it was shown that, in the BS model for example, the asymptotic state is characterized by a fractal set of active variables, with fractal dimension $d_f < 1$, for the one-dimensional BS model. This implies that the weight of active, correlated, variables vanishes in the thermodynamic limit as $L^{-(d-d_f)}$.

In this paper we showed, both by an analytical argument and by numerical simulations, that spatial correlations between variables in extremal models are a finite size effect which vanishes in the thermodynamic limit. This result supports the basic assumption made in the quenched-stochastic transformation [5,9] and suggests that this mapping is exact in the limit $L \to \infty$ of a large number of variables.

[1] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[2] D. Wilkinson and J. F. Willemsen, J. Phys. A **16**, 3365 (1983).
[3] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992).
[4] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[5] M. Marsili, J. Stat. Phys. **77**, 733 (1994).
[6] R. Cafiero, A. Gabrielli, M. Marsili, and L. Pietronero, Phys. Rev. E **54**, 1406 (1996).
[7] M. Marsili, Europhys. Lett. **28**, 385 (1994).
[8] A. Erzan, L. Pietronero, and A. Vespignani, Rev. Mod. Phys. **67**, 545 (1995).
[9] A. Gabrielli, M. Marsili, R. Cafiero, and L. Pietronero, J. Stat. Phys. **84**, 889 (1996).
[10] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics* (John Wiley & Sons, New York, 1978).
[11] A. Schenkel, J. Zhang, and Y.-C. Zhang, Fractals **1**, 41 (1993).